

# NEW INTEGRABLE GENERALIZATION OF THE ONE-DIMENSIONAL t-J MODEL

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## Abstract

A new generalization of the t-J model with a nearest-neighbour hopping is formulated and solved exactly by the Bethe-ansatz method in the thermodynamic limit. The model describes the dynamics of fermions with different spins and with isotropic and anisotropic interactions.

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In recent years there has been considerable interest in studying low- dimensional electronic models of strong correlation due to the possibility that the normal state of the two-dimensional novel superconductivity may share some interesting features of a 1D interacting electron system [1]. In one dimension, the Bethe-ansatz technique can allow one to exactly solve Hamiltonians in special cases, such as the Hubbard model [2] and the ordinary  $t - J$  model at its supersymmetric point [3,4]. For example, it is possible to obtain the low-energy gapless excitation spectrum around the ground state by the finite-size scaling method [5,6] and calculate the critical exponents of the correlation functions [7-9].

The  $t - J$  model is a lattice model on the restricted electronic Hilbert space, where the occurrence of two electrons on the same lattice site is forbidden. This restriction corresponds to an implicitly infinite on-site Coulomb repulsion. Two types of interactions between electrons on nearest-neighbours sites are considered: a charge interaction of strength  $V$  and a spin- exchange interaction  $J$ . The Hamiltonian of the extended version of the  $t - J$  model has the form [3,4,10]

$$\begin{aligned}
H = & - \sum_{j=1}^L \sum_{\alpha=1}^N P \left( c_{j,\alpha}^+ c_{j+1,\alpha} + c_{j+1,\alpha}^+ c_{j,\alpha} \right) P \\
& - \sum_{j=1}^L \left[ J \sum_{\alpha \neq \beta}^N c_{j,\alpha}^+ c_{j,\beta} c_{j+1,\beta}^+ c_{j+1,\alpha} + \sum_{\alpha,\beta=1}^N V_{\alpha\beta} c_{j,\alpha}^+ c_{j,\alpha} c_{j+1,\beta}^+ c_{j+1,\beta} \right] \quad (1)
\end{aligned}$$

where  $c_{j,\alpha}$  annihilates an electron with a spin component  $\alpha$ ,  $P$  is the projector on the subspace of non-doubly occupied states, and  $L$  is the lattice size. We have introduced an anisotropy in the charge interactions through a matrix  $V_{\alpha\beta}$ .

In the isotropic case  $V_{\alpha\beta} = V$  the Hamiltonian (1) corresponds to the traditional  $t - J$  model which was exactly solved by the Bethe-ansatz method at the supersymmetric point ( $V = -J = 1$ ) for the case  $S = 1/2$ . [3,4,9-11]. The generalization of this result for the arbitrary spin  $S$  was carried out in [12-14]. Other generalizations of the  $t - J$  model were studied in [15 -18]. In particular, in [18] the anisotropic generalization of the  $t - J$  model has been constructed and it was shown that the model (1) is solvable for the arbitrary spin and special values of the coupling  $J$  and  $V_{\alpha\beta}$ :

$$J = \epsilon_0,$$

$$V_{\alpha\beta} = \epsilon_0[(1 + \varepsilon_\alpha) \cosh \gamma \cdot \delta_{\alpha\beta} + \exp[\text{sign}(\alpha - \beta)\gamma](1 - \delta_{\alpha\beta})] \quad (2)$$

where  $\gamma > 0$  is a measure of the anisotropy. It was shown more exactly in [18] that the Hamiltonian (1,2) is the quantum counterpart of the so called Perk-Schultz model [19] which was diagonalized by Schultz in the most general form [20] (see also [21-22]). In fact, in paper [19] Perk and Schultz have considered two models, the Hamiltonian of the first one is given by (1,2) and the Hamiltonian of the second one is given by [19,22]

$$H = - \sum_{j=1}^L \sum_{\alpha, \beta=1}^N c_{j\alpha}^+ c_{j\beta} c_{j+1, \alpha}^+ c_{j+1, \beta}, \quad (3)$$

The systems which are described by (3) and their different modifications were studied in spin formulation in [23-25] (see also references therein). In this letter we present a new set of models of strongly-correlated particles which are exactly solvable. In the system there are different types of particles and interactions between two particles in the neighbour sites are given by either Hamiltonian (1) or Hamiltonian (3) and depend on the type of particles. Thus we will consider the system the Hamiltonian of which contains the interactions of both types (1) and (3).

Firstly let us formulate the problem and write down the Hamiltonian. Consider a periodic one-dimensional lattice of  $L$  sites. Place  $n$  particles on this lattice and specify its nature. We assume  $P$  components, denoted  $Q_1, Q_2, \dots, Q_P$  and define  $n_{Q_i}$  as the number of particles of component  $Q_i$

$$(\sum_{i=1}^P n_{Q_i} = n).$$

In each set there is  $N_{Q_i}$

$$(\sum_{i=1}^P N_{Q_i} = N).$$

types of particles, so that the summation

$$\sum_{\alpha \in Q_i} \text{ means } \sum_{\alpha = N_{Q_1} + \dots + N_{Q_{i-1}} + 1}^{N_{Q_1} + \dots + N_{Q_i}}.$$

We establish the one-to-one correspondence between particles  $\alpha \Leftrightarrow \bar{\alpha}(\alpha, \bar{\alpha} \in Q_j)$ . So in each set there are  $q_{Q_j}$  conjugate pairs of particles

$$[(N_{Q_i} + 1)/2] \leq q_{Q_i} \leq N_{Q_i}$$

where  $[N/2]$  means the integer part of the number  $N/2$ . The dynamics of above defined particles is described by the Hamiltonian

$$\begin{aligned} H = & - \sum_{j=1}^L \sum_{\alpha=1}^N P \left( c_{j,\alpha}^+ c_{j+1,\alpha} + c_{j+1,\alpha}^+ c_{j,\alpha} \right) P \\ & - \sum_{j,i} \sum_{\alpha, \beta \in Q_i} \epsilon_0 \epsilon_i [U_\alpha^{(i)} U_\beta^{(i)} c_{j,\alpha}^+ c_{j\beta} c_{j+1,\bar{\alpha}}^+ c_{j+1,\bar{\beta}} - (1 + \epsilon_i) \cosh \gamma n_{j\alpha} n_{j+1,\beta}] \\ & - \sum_{j,i \neq k} \sum_{\alpha \in Q_i, \beta \in Q_k} \epsilon_0 \{ g_{ik} c_{j,\alpha}^+ c_{j\beta} c_{j+1,\beta}^+ c_{j+1,\alpha} - \exp[\text{sign}(\alpha - \beta)\gamma] n_{j\alpha} n_{j+1,\beta} \}, \end{aligned} \quad (4)$$

where  $\epsilon_i, \epsilon_0 = \pm 1$  and the parameters  $U_\alpha^{(i)}$  play the role of anisotropies inside of each set  $Q_i$  and satisfy

$$U_\alpha^{(i)} = 1/U_{\bar{\alpha}}^{(i)}, \quad \sum_{\alpha \in Q_i} (U_\alpha^{(i)})^2 = 2 \cosh \gamma \quad (5)$$

$$g_{ij} = g_{ji}^{-1}.$$

The ordinary  $t - J$  model given in (1,2) is obtained by choosing sector of (4) in which there are particles only of one type from each set  $Q_j$  and for this type  $\alpha \neq \bar{\alpha}$ . The anisotropic version of the Perk-Schultz model of the second type (3) is obtained by choosing a sector in which we have particles only of one component, for example,  $Q_1$  and  $N = L$ . Previously we solved exactly the model (4) at this case also for  $N < L$  [25,26] for the different choice of conjugate pairs. The model which is described by the Hamiltonian (4) is also the generalization of the model considered by Sutherland [10] for the case when particles of the a given component are not identical and the anisotropy has been introduced in the system. In the general case the Hamiltonian (4) describes the dynamics of fermions with different spins. For example, the case  $N_{Q_i} = 2$  corresponds to  $S = 1/2$ , the case  $N_{Q_i} = 3$  corresponds to  $S = 1$  with biquadratic interactions [26]. For the general case  $S_i = (N_{Q_j} - 1)/2$  the magnetic interactions inside of each set can be written as a polynomial of

degree  $2S_i$  in the spin operator. Moreover the interactions between particles of different sets also exist.

The exact solution for the eigenstates and eigenvalues of the Hamiltonian (4) can be obtained within the framework of the Bethe-ansatz method [27,28]. The central object of this method is the two-particle scattering matrix  $S$  which is calculated from the single- and two-particle processes described by the Hamiltonian (4). The nonvanishing elements of the S-matrix are

$$\begin{aligned}
S_{\alpha'\beta'}^{\alpha\beta}(k_1, k_2) &= [\sin(i\gamma - \lambda_1 + \lambda_2)]^{-1} \hat{S}_{\alpha'\beta'}^{\alpha\beta}(\lambda_1 - \lambda_2); \\
\hat{S}_{\alpha'\beta'}^{\alpha\beta}(\lambda) &= \delta_{\alpha\beta'} \delta_{\beta, \alpha'} \sin(i\gamma + \varepsilon_i \lambda) \\
&\quad - \varepsilon_i \delta_{\alpha\bar{\beta}} \delta_{\beta' \bar{\alpha}'} U_{\alpha}^{(i)} U_{\beta'}^{(i)} \sin \lambda; \text{ for } \alpha, \beta \in Q_i \\
\hat{S}_{\alpha'\beta'}^{\alpha\beta}(\lambda) &= i \delta_{\alpha\beta'} \delta_{\beta, \alpha'} \sinh \gamma \exp[i \text{sign}(\beta - \alpha) \lambda] \\
&\quad - G_{ik} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \sin \lambda; \text{ for } \alpha \in Q_i; \beta \in Q_k, i \neq k
\end{aligned} \tag{6}$$

where  $\lambda_j$  ( $j = 1, 2, \dots, n$ ) are suitable particle rapidities related to the momenta  $\{k_j\}$  of the electrons by

$$k_j = \begin{cases} \pi - \Theta(\lambda_j; \frac{1}{2}\gamma), & \epsilon_0 = -1, \\ -\Theta(\lambda_j; \frac{1}{2}\gamma), & \epsilon_0 = +1, \end{cases} \tag{7}$$

with the function  $\Theta$  defined by

$$\Theta(\lambda; \gamma) = 2 \arctan(\cot \gamma \cdot \tan \lambda); \quad -\pi < \Theta(\lambda, \gamma) \leq \pi. \tag{8}$$

A necessary and sufficient condition for the applicability of the Bethe-ansatz method is the Yang-Baxter equations [27,29]. Up to our knowledge the form of the S-matrix (6) is a new one, therefore it is necessary to check the Yang-Baxter equations. We have checked these equations numerically for the different choice of parameters  $U_{\alpha}^{(i)}$  and the sets  $Q_i$ . We may note also that  $\hat{S}$ -matrix (6) can be obtained by the baxterization of the following Hamiltonian

$$\begin{aligned}
H &= \sum_{j=1}^{N-1} e_j; \\
e_j &= \cosh \gamma + \sum_{i \neq k} \sum_{\alpha \in Q_i, \beta \in Q_k} [G_{ik} E_j^{\alpha\beta} E_{j+1}^{\beta\alpha} - \sinh \gamma \text{sign}(\alpha - \beta) E_j^{\alpha\alpha} E_{j+1}^{\beta\beta}], \\
&\quad + \sum_i \sum_{\alpha, \beta \in Q_i} \varepsilon_1^{(i)} [U_{\alpha}^{(i)} U_{\beta}^{(i)} E_j^{\alpha\beta} E_{j+1}^{\bar{\alpha}\bar{\beta}} - \cosh \gamma E_j^{\alpha\alpha} E_{j+1}^{\beta\beta}];
\end{aligned} \tag{9}$$

where the  $N \times N$  matrices  $E^{\alpha\beta}$  have elements

$$(E^{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}$$

The quantum chain (9) have important property that if  $e_j$  satisfies the following Hecke algebra [31]

$$\begin{aligned} e_j e_{j\pm 1} e_j - e_j &= e_{j\pm 1} e_j e_{j\pm 1} - e_{j\pm 1} \\ [e_j, e_k] &= 0, \text{ for } |j - k| \geq 2 \\ e_j^2 &= 2 \cosh \gamma e_j \end{aligned} \quad (10)$$

then the S-matrix (6) satisfies the Yang-Baxter equations and the model (4) is integrable [31]. We have checked that  $e_j$  satisfies the equations (10) and thus we have proved the integrability of the model under consideration (4).

The Hamiltonian (4) is diagonalized by a standard procedure by imposing periodic boundary conditions on the Bethe function. These boundary conditions can be expressed in terms of the transfer matrix of the non-uniform models (6) by using the quantum method of the inverse problem [31,32].

The rapidities  $\lambda_j$  that define a n-particle wave function are obtained by solving the equations

$$\left[ \frac{\sinh(\lambda_j - i\gamma/2)}{\sinh(\lambda_j + i\gamma/2)} \right]^L = (-1)^{n-1} \Lambda(\lambda_j), \quad (11)$$

where  $\Lambda(\lambda)$  is the eigenvalue of the transfer matrix

$$T_{\{\alpha'_i\}}^{\{\alpha_i\}}(\lambda) = \sum_{\{\beta_i\}} \prod_{l=1}^n S_{\alpha'_l \beta_l}^{\alpha_l \beta_{l+1}}(\lambda_l - \lambda), \quad (\beta_{n+1} = \beta_1). \quad (12)$$

It is simple to verify that besides the numbers of particles in each set  $n_{Q_i}$  the numbers of "conjugate" pairs in each set are conserved quantities in the Hamiltonian (4). Here we denote two conjugate particles of same set paired if they are consecutive particles and have no unpaired particles of this set between them.

In a general case, the complete diagonalization of the transfer matrix (12) is not a simple problem even in the simplest special cases. Therefore here we restrict ourself to consideration of the thermodynamic limit of the Hamiltonian (4), when the periodic and free boundary conditions are identical.

Firstly consider model (4) in the sector where we have no pairs of particles. If we have no pairs of a set  $Q_i$  then the first interaction term in the Hamiltonian (4) does not work and all particles of this set are identical and can be considered as one component of the model. In this way the general model (4) in this sector can be reduced to the anisotropic  $t - J$  model with  $P$  components [18,20] and the diagonalization of the transfer matrix of the inhomogeneous model (12) gives the following Bethe-ansatz equations

$$\prod_{j'=1}^{m_{\sigma-1}} \frac{\sin(\lambda_j^{(\sigma)} - \lambda_{j'}^{(\sigma-1)} + \frac{i}{2}\epsilon_{\sigma}\gamma)}{\sin(\lambda_j^{(\sigma)} - \lambda_{j'}^{(\sigma-1)} - \frac{i}{2}\epsilon_{\sigma}\gamma)} = -\epsilon_{\sigma}^{n_{\sigma}} \epsilon_{\sigma+1}^{n_{\sigma+1}} \prod_{\rho=0}^q (G_{\sigma\rho} G_{\rho\sigma+1})^{n_{\rho}} \\ \times \prod_{j'=1}^{m_{\sigma}} \frac{\sin(\lambda_j^{(\sigma)} - \lambda_{j'}^{(\sigma)} + i\epsilon_{\sigma+1}\gamma)}{\sin(\lambda_j^{(\sigma)} - \lambda_{j'}^{(\sigma)} - i\epsilon_{\sigma}\gamma)} \prod_{j'=1}^{m_{\sigma+1}} \frac{\sin(\lambda_j^{(\sigma)} - \lambda_{j'}^{(\sigma+1)} - \frac{i}{2}\epsilon_{\sigma+1}\gamma)}{\sin(\lambda_j^{(\sigma)} - \lambda_{j'}^{(\sigma+1)} + \frac{i}{2}\epsilon_{\sigma+1}\gamma)}; \quad (13)$$

where

$$\begin{aligned} j &= 1, 2, \dots, m_{\sigma}; \quad \sigma = 0, 1, \dots, q-1. \\ \lambda_j^{(-1)} &= 0; \quad \lambda_j^{(0)} = \lambda_j; \\ n_j &= m_{j-1} - m_j; \quad m_1 = L; \quad m_q = 0; \quad n_o = n \end{aligned} \quad (14)$$

and

$$\begin{aligned} q &= P; \quad \epsilon_0 = 1; \quad G_{\sigma\sigma} = G_{0\sigma} = G_{\sigma 0} = 1; \\ \epsilon_i &= \varepsilon_i; \quad G_{ik} = g_{ik}; \quad (i, k = 1, 2, \dots, P) \end{aligned} \quad (15)$$

$n_j$  is the number of particles from the set  $Q_j$ .

The total energy and momentum of the model are given in terms of the particle rapidities  $\lambda_j$  in the following form

$$\begin{aligned} E &= -2 \sum_{j=1}^n \cos k_j = 2\varepsilon\varepsilon_1 \sum_{j=1}^n \left( \cosh \gamma - \frac{\sinh^2 \gamma}{\cosh \gamma - \cos 2\lambda_j} \right), \\ P &= \sum_{j=1}^n k(\lambda_j). \end{aligned} \quad (16)$$

Consider now the model (4) in the general case when in each sector  $Q_j$  we have  $n_j$  particles and  $n'_j$  "conjugate" pairs of particles. The reference

state  $\Psi_0$  is made up of a state in which there are no "conjugate" pairs. Examining (4) we see that when this Hamiltonian acts on a state, it looks for "conjugate" pairs and replaces them by a sum over all such pairs from a given set  $Q_j$ . The second possible process is the permutation of two neighbour particles or one particle and one conjugate pair. It is important that both processes do not depend on the number of types of particles in each set  $N_{Q_j}$  up to the boundary. Thus the Bethe-ansatz equations depend only on  $n_j$  and  $n'_j$ . Therefore we may consider the Hamiltonian (4) for the case when all  $N_{Q_j} = 2$ . Certainly the fact that the solution in the general case is strongly in line with the solution for the case  $N_{Q_j} = 2$  is connected with the Hecke algebra. The two models with open boundary conditions have the same energy levels with different multiplications if they have the same underlying Hecke algebra [30]. Thus we consider the Hamiltonian (4) in the thermodynamic limit when the periodic and open boundary conditions are equivalent for the case when all  $N_{Q_j} = 2$ . The Bethe-ansatz equations for this case will be valid for the arbitrary choice of  $N_{Q_j}$ . The examination of the case  $N_{Q_j} = 2$  shows that the Hamiltonian (4) can be reduced directly to the 2P-component  $t - J$  model [18,20]. Thus the general solution of the model (4) is given by (13-14) and (16) with redefined parameters  $\varepsilon_\sigma$  and  $G_{\sigma\rho}$

$$\begin{aligned}
q &= 2P; \quad \epsilon_0 = 1; \quad G_{\sigma\sigma} = G_{0\sigma} = G_{\sigma 0} = 1; \\
\epsilon_i &= \varepsilon_{[(i+1)/2]}; \quad G_{ik} = g_{[(i+1)/2],[(k+1)/2]}; \quad [(i+1)/2] \neq [(k+1)/2]; \\
&\quad (i, k = 1, 2, \dots, 2P); \\
G_{2l-1,2l} &= G_{2l,2l-1} = \varepsilon_l; \quad (l = 1, 2, \dots, P) \quad (17)
\end{aligned}$$

and now  $n_{2l-1}$  and  $n_{2l} - n_{2l-1}$  are numbers of "conjugate" pairs and separate particles of the set  $Q_l (l = 1, 2, \dots, P)$ .

The ground state energy and the excited states of the Hamiltonian (4) can be calculated in principle by straightforward methods on the base of Bethe-ansatz equations (13-14) and (17), which in the thermodynamic limit can be written as integral equations. However, the solution of these equations depends strongly on sign functions  $\epsilon_\alpha$  and the phase factors  $g_{ik}$  and can be subject of separate considerations for the different choice of these parameters.



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